Some variational inequality in $L^r$ and its application to the Helmholtz-Weyl decomposition in 3-D bounded domains.

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Let us first impose the following assumption on the domain $\Omega$:

Assumption. $\Omega$ is a bounded domain in $\mathbb{R}^3$ with the $C^{2+\mu}$-boundary $\partial\Omega$, where $\mu > 0$.

We denote by $C^\infty_0(\Omega)$ the set of all $C^\infty$-vector functions $\varphi = (\varphi^1, \varphi^2, \varphi^3)$ with compact support in $\Omega$, such that $\text{div}\ \varphi = 0$. $L^r_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ with respect to the $L^r$-norm $\|\cdot\|_r$; $(\cdot, \cdot)$ denotes the duality pairing between $L^r(\Omega)$ and $L^r'(\Omega)$, where $1/r + 1/r' = 1$. $L^r(\Omega)$ stands for the usual (vector-valued) $L^r$-space over $\Omega$, $1 < r < \infty$. Let us define the space $V^r(\Omega)$ by

\[ V^r(\Omega) \equiv \{ u \in L^r(\Omega); \text{div}\ u \in L^r(\Omega), \text{rot}\ u \in L^r(\Omega), u \times \nu|_{\partial\Omega} = 0 \}, \quad 1 < r < \infty. \]

It is easy to see that if $u \in L^r(\Omega)$ with rot $u \in L^r(\Omega)$, then it holds $u \times \nu \in W^{1-1/r', r'}(\partial\Omega)^*$. Equipped with the norm $\|u\|_{V^r}$

\[ \|u\|_{V^r} \equiv \|\text{div}\ u\|_r + \|\text{rot}\ u\|_r + \|u\|_r, \]

we may regard $V^r(\Omega)$ as a closed subset of $W^{1,r}(\Omega)$. Indeed, we have that $V^r(\Omega) \subset W^{1,r}(\Omega)$ with

\[ \|\nabla u\|_r \leq C\|u\|_{V^r} \quad \text{for all} \ u \in V^r(\Omega), \]

where $C = C(r)$ is a constant depending only on $r$. Furthermore, we define $V^r_\sigma(\Omega)$ by

\[ V^r_\sigma(\Omega) \equiv \{ u \in V^r(\Omega); \text{div}\ u = 0 \ \text{in} \ \Omega \}. \]

Finally, we denote by $\mathcal{H}(\Omega)$ the space of harmonic vector fields on $\Omega$

\[ \mathcal{H}(\Omega) \equiv \{ h \in C^\infty(\Omega) \cap C^2(\Omega); \text{div}\ h = 0, \text{rot}\ h = 0 \ \text{in} \ \Omega, \ h \cdot \nu|_{\partial\Omega} = 0 \}. \]

It is well-known that the dimension of $\mathcal{H}(\Omega)$ is finite. For more precise characterization of $\mathcal{H}(\Omega)$, see Remark 1 (2) below.

Our main result now reads

Theorem 1 Let $\Omega$ be as in the Assumption. Suppose that $1 < r < \infty$. Then for every $u \in L^r(\Omega)$, there are $p \in W^{1,r}(\Omega)$, $w \in V^r_\sigma(\Omega)$ and $h \in \mathcal{H}(\Omega)$ such that $u$ can be represented as

\[ u = h + \text{rot}\ w + \nabla p. \]
Such a triplet \( \{p, w, h\} \) is subordinate to the estimate

\[
\|\nabla p\|_r + \|w\|_{V_r^+} + \|h\|_r \leq C\|u\|_r
\]

with the constant \( C = C(r) \) independent of \( u \). The above decomposition (0.4) is unique. In fact, if \( u \) has another expression

\[
u = \tilde{h} + \text{rot } \tilde{w} + \nabla \tilde{p}
\]

for \( \tilde{h} \in H(\Omega), \tilde{w} \in V^r_\sigma(\Omega) \) and \( \tilde{p} \in W^{1,r}(\Omega) \), then we have

\[
h = \tilde{h}, \quad \text{rot } w = \text{rot } \tilde{w}, \quad \nabla p = \nabla \tilde{p}.
\]

An immediate consequence of the above theorem is

**Corollary 1** Let \( \Omega \) be as in the Assumption. By the unique decomposition (0.4) we have

\[
L^r(\Omega) = H(\Omega) \oplus \text{rot } V^r_\sigma(\Omega) \oplus \nabla W^{1,r}(\Omega), \quad 1 < r < \infty. \quad (\text{direct sum})
\]

Let \( S_r, R_r \) and \( Q_r \) be projection operators associated to (0.4) from \( L^r(\Omega) \) onto \( H(\Omega), \text{rot } V^r_\sigma(\Omega) \) and \( \nabla W^{1,r}(\Omega) \), respectively, i.e.,

\[
S_r u \equiv h, \quad R_r u \equiv \text{rot } w, \quad Q_r u \equiv \nabla p.
\]

Then we have

\[
\|S_r u\|_r \leq C\|u\|_r, \quad \|R_r u\|_r \leq C\|u\|_r, \quad \|Q_r u\|_r \leq C\|u\|_r
\]

for all \( u \in L^r(\Omega) \), where \( C = C(r) \) is the constant depending only on \( 1 < r < \infty \). Moreover, there holds

\[
\begin{align*}
S^2_r &= S_r, & S^*_r &= S^*_r, \\
R^2_r &= R_r, & R^*_r &= R^*_r, \\
Q^2_r &= Q_r, & Q^*_r &= Q^*_r,
\end{align*}
\]

where \( S^*_r, R^*_r \) and \( Q^*_r \) denote the adjoint operators on \( L^r(\Omega) \) of \( S_r, R_r \) and \( Q_r \), respectively.

**Remark 1.** (1) It is known that

\[
L^r(\Omega) = L^r_\sigma(\Omega) \oplus \nabla W^{1,r}(\Omega), \quad 1 < r < \infty, \quad (\text{direct sum}).
\]

See Fujiwara-Morimoto [4], Solonnikov [11] and Simader-Sohr [9]. Our decomposition (0.7) gives a more precise direct sum of \( L^r_\sigma(\Omega) \) such as

\[
L^r_\sigma(\Omega) = H(\Omega) \oplus \text{rot } V^r_\sigma(\Omega), \quad 1 < r < \infty. \quad (\text{direct sum})
\]

(2) Suppose that the boundary \( \partial \Omega \) has \( L + 1 \) connected components \( \Gamma_0, \Gamma_1, \cdots, \Gamma_L \) of \( C^2 \)-surfaces such that \( \Gamma_1, \cdots, \Gamma_L \) lie inside of \( \Gamma_0 \) with \( \Gamma_i \cap \Gamma_j = \phi \) for \( i \neq j \), and such that

\[
\partial \Omega = \bigcup_{j=0}^L \Gamma_j.
\]
Moreover, we assume that there are $N$ $C^2$-surfaces $\Sigma_1, \cdots, \Sigma_N$ such that $\Sigma_i \cap \Sigma_j = \phi$ for $i \neq j$, and such that

$$\hat{\Omega} \equiv \Omega \setminus \Sigma, \Sigma \equiv \bigcup_{j=1}^{N} \Sigma_j$$

is simply connected.

Then Foias-Temam [3] showed that

$$\dim \mathcal{H}(\Omega) = N.$$  \hspace{1cm} (0.15)

They [3] also gave an orthogonal decomposition of $L^2_\sigma(\Omega)$ such as

$$L^2_\sigma(\Omega) = \mathcal{H}(\Omega) \oplus H_1(\Omega)$$

(orthogonal sum in $L^2(\Omega)$),

where

$$H_1(\Omega) \equiv \{ u \in L^2_\sigma(\Omega); \int_{\Sigma_j} u \cdot \nu dS = 0, \ j = 1, \cdots, N \}.$$

Yoshida-Giga [13] investigated the operator $\text{rot}$ with its domain $D(\text{rot}) = \{ u \in H_1(\Omega); \text{rot } u \in H^1(\Omega) \}$ and showed that $H^1(\Omega)$ is simply connected. \hspace{1cm} (0.16)

Furthermore, they [13] gave another type of orthogonal $L^2$-decomposition of vector fields $u \in D(\text{rot})$. From our decomposition (0.12) with $r=2$, it follows also that $H^1(\Omega) = \text{rot } V^2(\Omega)$. \hspace{1cm} (3)

In the case when $\Omega$ is a star-shaped domain, Griesinger [5] gave a similar decomposition in $L^r(\Omega)$ for $1 < r < \infty$. In her case, it holds $N = 0$. Since she took the smaller space $W^{1,r}_0(\Omega)$ than our space $V^r(\Omega)$, it seems to be an open question whether, in the same way as in (0.7), the anihilator $\text{rot } W^{1,r}_0(\Omega)$ coincides with $\nabla W^{1,r}_0(\Omega)$.

As an application of our decomposition, we have the following gradient estimates of vector fields via $\text{div}$ and $\text{rot}$.

**Corollary 2** Assume that $1 < r < \infty$.

(1) Let $u \in L^r(\Omega)$ with $\text{div } u \in L^r(\Omega)$, $\text{rot } u \in L^r(\Omega)$ and $u \cdot \nu|_{\partial \Omega} = 0$. Then we have $u \in W^{1,r}(\Omega)$ with the estimate

$$\|\nabla u\|_r \leq C(\|\text{div } u\|_r + \|\text{rot } u\|_r + \|u\|_{1}), \hspace{1cm} (0.16)$$

where $C = C(r)$ is the constant independent of $u$.

(2) Let $u \in W^{s,r}(\Omega)$ for $s > 1 + 3/r$ with $u \cdot \nu|_{\partial \Omega} = 0$. Then we have $\nabla u \in L^\infty$ with the estimate

$$\|\nabla u\|_\infty \leq C \{ 1 + \|u\|_r + (\|\text{div } u\|_{\text{bmo}} + \|\text{rot } u\|_{\text{bmo}}) \log(e + \|u\|_{W^{s,r}}) \}, \hspace{1cm} (0.17)$$

where $C = C(r)$ is the constant independent of $u$. For definition of the bmo-norm, see Remark 2 below.

**Remark 2**. (1) Let us recall the bmo-norm in $\Omega$. For $f \in L^1_{\text{loc}}(\mathbb{R}^3)$, we define $\|f\|_{\text{bmo}}(\mathbb{R}^3)$ by

$$\|f\|_{\text{bmo}}(\mathbb{R}^3) = \sup_{x \in \mathbb{R}^3, 0 < R < 1} \frac{1}{|B_R(x)|} \int_{B_R(x)} |f(y) - f_{B_R(x)}| dy \sup_{x \in \mathbb{R}^3} \frac{1}{|B_1(x)|} \int_{B_1(x)} |f(y)| dy$$

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with \( f_{BR(x)} = \frac{1}{|BR(x)|} \int_{BR(x)} f(y) dy \), where \( BR(x) \) denotes the ball in \( \mathbb{R}^3 \) centered at \( x \) with radius \( R \) and \( |BR(x)| \) is its volume. For \( g \in L^1_{\text{loc}}(\Omega) \) we say \( g \in bmo(\Omega) \) if there is an extension \( f \in bmo(\mathbb{R}^3) \) such that \( g = f \) on \( \Omega \). The \( bmo \)-norm \( \|g\|_{bmo} \) of \( g \) on \( \Omega \) is defined by

\[
\|g\|_{bmo} \equiv \inf \{\|f\|_{bmo(\mathbb{R}^3)}; f \in bmo(\mathbb{R}^3), f = g \text{ on } \Omega\}.
\]

(2) von Wahl [12] proved that (0.16) without \( \|u\|_1 \) on the right hand side holds if and only if \( N = 0 \), i.e., \( \Omega \) is simply connected. He also showed the same estimate for \( u \in W^{1,r}(\Omega) \) with \( u \times \nu = 0 \) on \( \partial \Omega \) if and only if \( L = 0 \). Our variational inequality makes it possible to prove (0.16) also for \( u \in W^{1,r}(\Omega) \) with \( u \times \nu = 0 \) on \( \partial \Omega \). von Wahl’s estimate [12] may be regarded as a special case of ours since we can treat the general case such as (0.13) and (0.14). His method is based on the representation formula for \( u \in W^{1,r}(\Omega) \) via \( \text{div} \ u \) and \( \text{rot} \ u \) which is different from ours.

(3) In \( \mathbb{R}^3 \), by means of the Biot-Savard law, Beale-Kato-Majda [1] and Kozono-Taniuchi [6] obtained a similar estimate to (0.17) for \( u \in W^{s,r}(\mathbb{R}^3) \) with \( s > 1 + 3/r \). More generalized version in the homogeneous Besov space \( \dot{B}^{0,\infty}_{\infty,\infty}(\mathbb{R}^3) \) is found in Kozono-Ogawa-Taniuch [7]. In the case of simply connected bounded domains \( \Omega \) in \( \mathbb{R}^3 \), Ferrari showed (0.17) for \( \text{div} u = 0 \) with \( u \cdot \nu|_{\partial \Omega} = 0 \). More general case such as (0.13) and (0.14) was treated by Shirota-Yanagisawa [10] and Ogawa-Taniuchi [8].

References


