Abstract approach to Schrödinger evolution equations

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Let \( \{A(t); 0 \leq t \leq T\} \) be a family of closed linear operators in a complex Hilbert space \( X \). We are concerned with linear evolution equations of the form

\[
\text{(E)} \quad \frac{d}{dt} u(t) + A(t) u(t) = f(t) \quad \text{on} \quad (0, T).
\]

Let \( S \) be a self-adjoint operator in \( X \), satisfying \( (u, Su) \geq \|u\|^2 \) for \( u \in D(S) \). Then the square root \( S^{1/2} \) is well-defined. Put \( Y := D(S^{1/2}) \) and \( (u, v)_Y := (S^{1/2}u, S^{1/2}v) \), \( u, v \in Y \). Then \( Y \) is a Hilbert space with norm \( \|v\|_Y := (v, v)_Y^{1/2} \) embedded continuously and densely in \( X \). For \( \{A(t)\} \) and \( S \) assume that

1. There is \( \alpha \in L^1(0, T), \alpha \geq 0 \), such that
   \[
   |\text{Re}(A(t)v, v)| \leq \alpha(t) \|v\|^2, \quad v \in D(A(t)), \text{ a.a. } t \in (0, T).
   \]
2. \( Y \subset D(A(t)) \), a.a. \( t \in (0, T) \).
3. There is \( \beta \in L^1(0, T), \beta \geq \alpha \), such that
   \[
   |\text{Re}(A(t)u, Su)| \leq \beta(t) \|S^{1/2}u\|^2, \quad u \in D(S), \text{ a.a. } t \in (0, T).
   \]
4. \( A(\cdot) \in L^2(0, T; B(Y, X)) \).

Under the assumption stated above we can prove the following

**Theorem.** Let \( f(\cdot) \in L^2(0, T; X) \cap L^1(0, T; Y) \). Then there exists a unique strong solution \( u(\cdot) \) of (E) with \( u(0) = u_0 \in Y \) such that \( u(\cdot) \in H^1(0, T; X) \cap C([0, T]; Y) \).

In particular, if \( A(\cdot) \) is strongly continuous on \([0, T]\) to \( B(Y, X) \) and \( \alpha, \beta \) are constants, then Theorem has already been proved in Okazawa [1].

Now let \( n \in \mathbb{N} \). By introducing \( S := 1 + \Delta^{2n} + |x|^{4n} \) we can apply the above-mentioned theorem to the Cauchy problem for Schrödinger evolution equations:

\[
\text{(SE)} \quad i \frac{\partial u}{\partial t}(x, t) - (-\Delta_x + V(x, t)) u(x, t) = 0, \quad \text{a.a. } t \in (0, \infty)
\]
in \( L^2(\mathbb{R}^N) \). The assumption is satisfied under the following conditions:

(V0) \( V(\cdot, t) \in C^{2n}(\mathbb{R}^N) \), a.a. \( t \in (0, \infty) \).

(V1) There is \( g_0 \in L^2_{\text{loc}}(0, \infty) \) such that \( |V(x, t)| \leq g_0(t)(1 + |x|^{2n}) \).

(V2) There are \( g_j \in L^1_{\text{loc}}(0, \infty) \) (\( 1 \leq j \leq 2n \)) such that

\[
\begin{align*}
\sum_{|\alpha| = j} |D_x^\alpha V(x, t)| &\leq g_j(t)(1 + |x|^j), \quad (1 \leq j \leq n), \\
\sum_{|\alpha| = n} |D_x^n \Delta_x^{j-n} V(x, t)| &\leq g_j(t)(1 + |x|^j), \quad (n < j \leq 2n, j - n : \text{even}), \\
\sum_{|\alpha| = n+1} |D_x^n \Delta_x^{j-n+1} V(x, t)| &\leq g_j(t)(1 + |x|^j), \quad (n < j \leq 2n, j - n : \text{odd}).
\end{align*}
\]

**References**