Holomorphic families of linear $m$-accretive operators in Banach spaces and application to Schrödinger operators in $L^p$

Yoshiki Maeda
(Tokyo University of Science)

For two closed linear operators $T$ and $A$ in a Banach space $X$ we consider

$$ T + \kappa A \text{ with domain } D_0 := D(T) \cap D(A), $$

where $\kappa$ is a complex parameter and $D_0$ is assumed to be non-trivial.

**Definition.** \{$(T(\kappa); \kappa \in G_0 \subset \mathbb{C})$\} is said to be a holomorphic family of type $(A)$ if 

(i) $T(\kappa)$ is a closed linear operator with domain $D(T(\kappa)) = D$ independent of $\kappa$.

(ii) $T(\kappa)u$ is holomorphic with respect to $\kappa \in G_0$ for every $u \in D$.

Now we consider the Schrödinger type operators $-\Delta + \kappa V(x)$ with $V(x) \geq 0$. Here $T := -\Delta$ with domain $D(T) := W^{2p}(\mathbb{R}^N)$ is $m$-accretive in $L^p = L^p(\mathbb{R}^N)$ ($1 < p < \infty$, $N \in \mathbb{N}$). Let $A$ be the maximal operator of multiplication by $V(x) \in L^p_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$: $Av(x) := V(x)u(x)$ with domain $D(A) := \{u \in L^p; Vu \in L^p\}$. Then $A$ is also $m$-accretive in $L^p$ and its Yosida approximation is given by $A_\varepsilon v(x) = V_\varepsilon(x)v(x)$ ($v \in L^p$), where $V_\varepsilon(x) := V(x)[1 + \varepsilon V(x)]^{-1}$ for $\varepsilon > 0$.

The nonnegative potential $V(x)$ is assumed to satisfy either (V) or $(V)_\varepsilon$:

(V) $V \in C^1(\mathbb{R}^N)$ and there are nonnegative constants $a$, $b$, and $c$ such that

$$ |\nabla V(x)|^2 \leq a[V(x)]^3 + b[V(x)]^2 + c[V(x)] \quad \forall x \in \mathbb{R}^N. $$

(V)$_\varepsilon$ is nothing but condition (V) in which $V(x)$ is replaced by $V_\varepsilon(x)$.

Then, generalizing Kato [1, Theorem 7.1], we have the following

**Theorem.** Let $T$ and $A$ be as stated above. Assume that (V) (or $(V)_\varepsilon$). Then

(i) \{$(T + \kappa A; \kappa \notin \Omega)$\} = \{-$\Delta + \kappa V(x); \kappa \notin \Omega$\} forms a holomorphic family of type $(A)$,

(ii) $T + \kappa A = -\Delta + \kappa V(x)$ is $m$-accretive in $L^p$ for $\kappa \notin \Omega$ with $\text{Re}\kappa \geq 0$, where

$$ \Omega : y^2 \leq \frac{p^2}{2(p-1)} \left( x - \frac{p-1}{4} a \right) \left( x - \frac{p-1}{8} a \right) \quad \text{and} \quad x \leq \frac{p-1}{4} a \quad (x + iy \in \mathbb{C}). $$

**Example.** We consider several typical examples.

(i) Let $V(x) := |x|^2$. Then $a = b = 0$ and $c = 4$ in (1). Thus we see from Theorem that \{\{-$\Delta + \kappa |x|^2; \kappa \notin \Omega$\}\} forms a holomorphic family of type $(A)$ and $-\Delta + \kappa |x|^2$ is $m$-accretive in $L^p$ for $\kappa \notin \Omega$ with $\text{Re}\kappa \geq 0$, where $\Omega$ is given by a sector region of the complex plane.

(ii) Let $V(x) := |x|^{-2}$ and so $V_\varepsilon(x) = (|x|^2 + \varepsilon)^{-1}$. Then $a = 4$ and $b = c = 0$ in $(V)_\varepsilon$. Thus we see from Theorem that \{-$\Delta + \kappa |x|^{-2}; \kappa \notin \Omega$\} forms a holomorphic family of type $(A)$ and $-\Delta + \kappa |x|^{-2}$ is $m$-accretive in $L^p$ for $\kappa \notin \Omega$ with $\text{Re}\kappa \geq 0$, where $\Omega$ is given by a hyperbolic region of the complex plane. However, this result is not sharp. We can obtain a sharper result for the inverse square potential $|x|^{-2}$ which include Kato’s result (see [1, Example 7.4]). Roughly speaking, instead of (2) we can obtain the closed hyperbolic region $y^2 \leq (x - C_1(p, N))(C_3(p)x - C_2(p, N))$ and $x \leq C_1(p, N) (x + iy \in \mathbb{C})$, where $C_j(p, N)$ ($j = 1, 2$) are constants dependent on $p$ and $N$, and $C_3(p) \geq 0$ is a constant dependent on $p$.

**References**