On the global existence of spatially periodic solutions to a class of complex Ginzburg-Landau equations

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In this talk we consider the Cauchy problem for a class of complex Ginzburg-Landau equations

\[ \begin{cases}
\frac{\partial u}{\partial t} = (\delta_1 + i\delta_2)\Delta u - i\mu |u|^{2\sigma} u, & (t, x) \in (0, \infty) \times \mathbb{R}^d, \\
u(0, x) = u_0(x), & x \in \mathbb{R}^d,
\end{cases} \]  

where \( i = \sqrt{-1}, \sigma > 0, \delta_1 > 0, \delta_2, \mu \in \mathbb{R} \) and \( d \in \mathbb{N} \). We discuss the existence and uniqueness of global solutions to (CGL) with initial value \( u_0 \in X_1^+(\mathbb{R}^d) \), where \( X_1^+(\mathbb{R}^d) \) is the Sobolev space of spatially periodic functions defined as follows:

\[ X_1^+(\mathbb{R}^d) := \left\{ u \in W^{1,1}_\text{loc}(\mathbb{R}^d); \ u(\cdot + n) = u(\cdot) \text{ for all } n \in \mathbb{Z}^d \right\}, \]

\[ \| u \|_{m,p} := \left( \sum_{|\alpha| \leq m} \int_{(0,1)^d} |D^\alpha u(x)|^p dx \right)^{1/p} \quad (1 \leq p < \infty), \quad \| u \|_{m,\infty} := \max_{|\alpha| \leq m} (\text{ess. sup}_{x \in (0,1)^d} |D^\alpha u(x)|). \]

**Definition.** A function \( u \) is said to be a **global solution** to (CGL) if (i) and (ii) are satisfied:

(i) \( u \in C([0, \infty); X_1^+(\mathbb{R}^d)) \cap C((0, \infty); X_2^+(\mathbb{R}^d)) \cap C^1((0, \infty); X_2^+(\mathbb{R}^d)) \);

(ii) \( u \) satisfies (CGL) on \((0, \infty)\) in \( X_2^+(\mathbb{R}^d) \).

Gao and Wang [1] established the existence and uniqueness of global solutions to (CGL) in the \( d \)-dimensional torus \( \mathbb{T}^d \). If we regard functions on \( \mathbb{T}^d \) as periodic functions on \( \mathbb{R}^d \), then their result is translated as follows:

**Gao-Wang’s result.** Let \( \delta_1 > 0, \delta_2 \in \mathbb{R} \) and \( \sigma \in \mathbb{N} \). Assume that

\[
2 - \frac{2}{\sqrt{1 + (\delta_2/\delta_1)^2} + 1} < p < 2 + \frac{2}{\sqrt{1 + (\delta_2/\delta_1)^2} - 1} \quad \text{and} \quad p > \sigma d.
\]

Then for \( u_0 \in X_1^+(\mathbb{R}^d) \) there exists a unique global solution to (CGL).

We focus our eyes on the case \( p = 2 \). In this case, \( \sigma \) and \( d \) satisfy (\#) only when \( \sigma = d = 1 \). Namely, Gao and Wang have not dealt with the case \( d \geq 2 \) or \( \sigma \neq 1 \).

The purpose of this talk is to relax the second condition in (\#) when \( p = 2 \) and to extend the restriction from \( \sigma \in \mathbb{N} \) to \( \sigma > 0 \). Assuming further that \( \delta_2 \mu > 0 \), we can obtain the global existence and uniqueness of solutions to (CGL) even when \( d \geq 2 \) or \( \sigma \neq 1 \).

**Main Theorem.**

Let \( \delta_1 > 0, \delta_2 \mu > 0 \) and

\[ 0 < \sigma < \infty \quad \text{if} \quad (d = 1, 2), \quad 0 < \sigma < \frac{1}{d - 2} \quad \text{if} \quad (d \geq 3). \]

Then for \( u_0 \in X_2^+(\mathbb{R}^d) \) there exists a unique global solution to (CGL).

**References**