**Brezis-Gallouet-Wainger type inequality and its application to the Navier-Stokes equations**

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This is a joint work with Kohei Nakao (Shinshu University). Let $\Omega$ be $\mathbb{R}^n$, $\mathbb{R}^n_+$, a bounded domain, or an exterior domain with $\partial\Omega \in C^\infty$. The motion of a viscous incompressible fluid in $\Omega$ is governed by the Navier-Stokes equations:

$$\text{(N-S)} \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla \pi = 0, & \text{div } u = 0 \qquad t \in (0,T), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, & u|_{t=0} = a, \end{cases}$$

where $u = (u^1(x,t), u^2(x,t), \cdots, u^n(x,t))$ and $\pi = \pi(x,t)$ denote the velocity vector and the pressure, respectively, of the fluid at the point $(x,t) \in \Omega \times (0,T)$ and $a$ is a given initial velocity. In this talk, we consider Serrin type regularity criteria of solutions to the 3-D Navier-Stokes equations. Let $p \geq 3$. It is known that if strong $L^p$-solutions $u$ of the Navier-Stokes equations on $(0,T)$ satisfies

$$(S) \qquad \int_0^T \|u\|^2_{L^\infty(\Omega)} d\tau < \infty,$$

then $u$ can be continued to the strong $L^p$-solution on $(0,T')$ for some $T' > T$. In this talk, we shall slightly relax the condition (S).

For this purpose, we use the Brezis-Gallouet-Wainger type inequality:

$$(BGW)_\beta \qquad \|u\|_{L^\infty} \leq C(1 + \|f\|_X \log^\beta(e + \|f\|_Y)).$$

Brezis-Gallouet-Wainger [2, 3] proved $(BGW)_\beta$ in the case $\beta = 1 - 1/p$, $X = W^{n/p,p}(\mathbb{R}^n)$, $Y = W^{n/q+\alpha,q}(\mathbb{R}^n)(\subset \dot{C}^\alpha)(\alpha > 0)$. Engler [5] proved the same inequality for general domains $\Omega$ if $n/p$ is an integer. Ozawa [16] also proved it for general domains $\Omega$ without any condition on $n/p$. When $\Omega = \mathbb{R}^n$, in [9], $(BGW)_\beta$ was proved for $0 \leq \beta \leq 1$, $X = B^0_{\infty,1/(1-\beta)}(\mathbb{R}^n)$ and $Y = C^\alpha(\mathbb{R}^n)$. By using the method given in [16], when $\Omega$ is a bounded domain, in [15], $(BGW)_\beta$ was proved for $\beta = 1$, $X = bmo(\Omega)$ and $Y = \dot{C}^\alpha(\Omega)$. We note that in [1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20] several inequalities of Brezis-Gallouet-Wainger type were established in the case $Y = \dot{C}^\alpha$ or $Y \subset \dot{C}^\alpha$ . On the other hand, there are several choice of $X$. Then, we have one question.

**What is the largest normed space $X$ that satisfies $(BGW)_\beta$ with $Y = \dot{C}^\alpha(\Omega)$?**.

In this talk, we also consider this problem.

We introduce Banach spaces of Morrey type and Besov type which are wider than $L^\infty$.

<u>Definition.</u> (1) (Morrey type space)

- $M_\beta(\Omega) := \{f \in L^1_{loc}(\bar{\Omega}); \|f\|_{M_\beta} < \infty\}$ is introduced by the norm

$$\|f\|_{M_\beta(\Omega)} := \sup_{x\in\Omega,\ 0<t<1} \frac{1}{|B(x,t)| \log^\beta(e + \frac{1}{t})} \int_{B(x,t)\cap\Omega} |f(y)| dy.$$

- $\tilde{M}_\beta(\Omega)$ is defined by

$$\tilde{M}_\beta(\Omega) := \overline{BC(\bar{\Omega})}^{\|\cdot\|_{M_\beta(\Omega)}}.$$

(2) (Modified Vishik's space). Let $\psi$ be a smooth function on $\mathbb{R}^n$ with $\hat{\psi}(\xi) = 1$ in $B(0,1/2)$ and $\hat{\psi}(\xi) = 0$ in $B(0,1)^c$. Then,

- $V_\beta(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n); \|f\|_{V_\beta} < \infty\}$ is introduced by the norm

$$\|f\|_{V_\beta} := \sup_{N=1,2,\cdots} \frac{\|\psi_N * f\|_\infty}{N^\beta}, \quad \text{where} \quad \psi_N(x) := 2^{nN}\psi(2^N x).$$

- $\tilde{V}_\beta$ is defined by

$$\tilde{V}_\beta := \overline{BUC(\mathbb{R}^n)}^{\|\cdot\|_{V_\beta}}.$$

*Remark.* (a) We have $M_\beta(\Omega) \supset L^\infty(\Omega)$ and $V_\beta(\mathbb{R}^n) \supset M_\beta(\mathbb{R}^n) \supset L^\infty(\mathbb{R}^n)$.

(b) $\tilde{V}_\beta(\mathbb{R}^n)$ and $\tilde{M}_\beta(\Omega)$ satisfy $(BGW)_\beta$. That is, if $\alpha \in (0,1)$ and $\beta > 0$, then there are constants $C_1, C_2 > 0$ such that

$$\|f\|_{L^\infty(\Omega)} \le C_2\Big\{1 + \|f\|_{M_\beta(\Omega)} \log^\beta\left(e + \|f\|_{\dot{C}^\alpha(\Omega)}\right)\Big\} \text{ for all } f \in \dot{C}^\alpha(\Omega) \cap \tilde{M}_\beta(\Omega) \text{ and}$$

$$\|f\|_{L^\infty(\mathbb{R}^n)} \le C_1\Big\{1 + \|f\|_{V_\beta(\mathbb{R}^n)} \log^\beta\left(e + \|f\|_{\dot{C}^\alpha(\mathbb{R}^n)}\right)\Big\} \text{ for all } f \in \dot{C}^\alpha(\mathbb{R}^n) \cap \tilde{V}_\beta(\mathbb{R}^n).$$

Now our results read as follows:

**Theorem 1.** *Let $\beta > 0$ and $X$ be a normed space. Assume that $X$ satisfies the following conditions (A):*

$$(A)\begin{cases} (1) & BC(\bar{\Omega}) \hookrightarrow X(\Omega) \subset L^1_{uloc}(\bar{\Omega}) \text{ and } BC(\bar{\Omega}) \text{ is dense in } X, \\ (2) & \|\cdot\|_X \text{ has a translation invariant property in the following sense} \\ & \qquad \big\|f(\cdot + y)\big|_\Omega\big\|_{X(\Omega)} \le \|f\|_{X(\Omega)} \text{ for all } y \in \mathbb{R}^n \text{ and all } f \in C_0(\Omega), \\ (3) & \|f\|_X \le \|g\|_X \text{ if } f, g \in BC(\bar{\Omega}) \text{ and } |f(x)| \le |g(x)| \text{ a.e. } x \in \Omega, \\ (4) & \text{there exist constants } \alpha \in (0,1) \text{ and } C > 0 \text{ such that} \\ & \qquad \|f\|_{L^\infty(\Omega)} \le C\Big\{1 + \|f\|_X \log^\beta\left(e + \|f\|_{\dot{C}^\alpha(\Omega)}\right)\Big\} \text{ for all } f \in \dot{C}^\alpha(\Omega) \cap X. \end{cases}$$

*Then $X$ is continuously embedded in $\tilde{M}_\beta(\Omega)$.*

**Remarks.** (i) Since $f(\cdot) = f(\cdot + y - y)$, (A-2) implies that $\|f\|_{X(\Omega)} = \|f(\cdot + y)\|_{X(\Omega)}$, if both of $f$ and $f(\cdot + y)$ belong to $C_0(\Omega)$.

(ii) Since $\tilde{M}_\beta$ satisfies $(A)$, Theorem 1 implies that $\tilde{M}_\beta$ is the largest normed space that satisfies conditions $(A)$.

If $\Omega = \mathbb{R}^n$, without the condition (A-3), we have a similar result as follows.

**Theorem 2.** *Let $\beta > 0$ and $X$ be a normed space. Assume that $X$ satisfies the following conditions (B):*

$$(B)\begin{cases} (1) & BUC(\mathbb{R}^n) \hookrightarrow X \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \text{ and } BUC \text{ is dense in } X, \\ (2) & \|\cdot\|_X \text{ has a translation invariant proprty, i.e.,} \\ & \qquad \|f(\cdot - y)\|_X = \|f\|_X \text{ for all } y \in \mathbb{R}^n, \\ (3) & \text{there exist constants } \alpha \in (0,1) \text{ and } C > 0 \text{ such that} \\ & \qquad \|f\|_{L^\infty(\mathbb{R}^n)} \le C\Big\{1 + \|f\|_X \log^\beta\left(e + \|f\|_{\dot{C}^\alpha(\mathbb{R}^n)}\right)\Big\} \text{ for all } f \in BC^\infty(\mathbb{R}^n). \end{cases}$$

*Then $X$ is continuously embedded in $\tilde{V}_\beta$.*

**Remark.** Since $\tilde{V}_\beta(\mathbb{R}^n)$ satisfies $(B)$, Theorem 2 implies that $\tilde{V}_\beta(\mathbb{R}^n)$ is the largest normed space that satisfies conditions $(B)$.

**Theorem 3.** *Let $n = 3$, $p \ge 3$, $a \in L^p_\sigma(\Omega) \cap \dot{W}^{1,2}_{0,\sigma}(\Omega)$ and $u$ be a solution to (N-S) on $(0,T)$ in the class*

$$S_p(0,T) := C([0,T); L^p_\sigma) \cap C^1((0,T); L^p_\sigma) \cap C((0,T); W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)).$$

*Assume that*

$$(A) \int_s^T \|u(\tau)\|^2_{M_{1/2}(\Omega)} d\tau < \infty \quad \text{for some } s \in (0,T).$$

*Then, $u$ can be continued to the solution in the class $S_p(0,T')$ for some $T' > T$.*

**Remark.** (i) When $\Omega = \mathbb{R}^3$, Condition (A) can be replaced by $\int_s^T \|u(\tau)\|^2_{V_{1/2}(\mathbb{R}^3)} d\tau < \infty$ for some $s \in (0, T)$.

(ii) We can also establish Beale-Kato-Majda type criteria.

# References


[1] Beale, J.T., Kato, T., Majda, A., Commun. Math. Phys. , **94** (1984), 61-66.

[2] Brezis, H., Gallouet, T., Nonlinear Anal. T.M.A. , **4** (1980), 677-681.

[3] Brezis, H., Wainger, S., Comm. Partial Differential Equations , **5** (1987), 773-789.

[4] Chemin, J-Y., Oxford Science Publications, 1998.

[5] Engler, H., Comm. Partial Differential equations. , **14** no. 4 (1989), 541-544.

[6] Ibrahim, S., Majdoub, M., Masmoudi, N., Proc. AMS., **135** (2007), 87-97.

[7] Ferrari, A. B., Commun. Math. Phys. , **155** (1993), 277-294.

[8] Kato, T., Ponce, G., Comm. Pure Appl. Math. , **41**, (1988) 891-907.

[9] Kozono, H., Ogawa, T., Taniuchi, Y., Math. Z. **242** (2002) 251-278.

[10] Kozono, H., Taniuchi, Y., Commun. Math. Phys. **214** (2000) 191-200

[11] Kozono, H., Wadade, H., Math. Z. **259** (2008) 935-950

[12] Matsumoto, T., Ogawa, T., Math. Nachr. **283** (2010) 1810-1828

[13] Morii, K., Sato, T., Wadade, H., Nonliniear Analysis **73** (2010) 1747-1766

[14] Ogawa, T., SIAM J. Math. Anal., **34** (2003), 1318-1330.

[15] Ogawa, T., Taniuchi, Y., J. Differential. Equation., **190** (2003), 39-63.

[16] Ozawa, T., J. Funct. Anal., **127** (1995), 259-269.

[17] Sawano, Y., Wadade, H., J. Fourier Anal. Appl. **19** (2013), 20-47.

[18] Shirota, T., Yanagisawa, T., Proc. Japan Acad., **69** Ser. A (1993) 77-82.

[19] Vishik, M., Ann. Sci. École Norm. Sup., **32** (1999) 769-812.

[20] Zajaczkowski, W. M., Bull. Polish Acad. of Sciences Mathematics **37** (1989), 169-181.