The Cauchy problem for the Finsler heat equation

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Abstract:
Let $N \geq 1$ and let $H \in C(\mathbf{R}^N) \cap C^1(\mathbf{R}^N \setminus \{0\})$ be a norm of $\mathbf{R}^N$, that is

$$\begin{cases} H \geq 0 \text{ in } \mathbf{R}^N \text{ and } H(\xi) = 0 \text{ if and only if } \xi = 0, \\ H \text{ is convex in } \mathbf{R}^N, \\ H(\alpha\xi) = |\alpha| H(\xi) \text{ for } \xi \in \mathbf{R}^N \text{ and } \alpha \in \mathbf{R}. \end{cases}$$

We denote by H_0 the dual norm of H defined by

$$H_0(x) := \sup_{\xi \in \mathbf{R}^N \setminus \{0\}} \frac{x \cdot \xi}{H(\xi)}.$$

Then

$$|x \cdot \xi| \leq H_0(x) H(\xi), \qquad H(\xi) = \sup_{x \in \mathbf{R}^N \setminus \{0\}} \frac{x \cdot \xi}{H_0(x)}.$$

For any $x \in \mathbf{R}^N$, $\xi \in \mathbf{R}^N$ and $R > 0$, we set

$$\begin{aligned} B_H(\xi, R) :=& \{\eta \in \mathbf{R}^N \,:\, H(\eta - \xi) < R\}, \\ B_{H_0}(x, R) :=& \{y \in \mathbf{R}^N \,:\, H_0(y - x) < R\}. \end{aligned}$$

We assume that

$$B_H(0,1) \text{ is strictly convex,}$$

which is equivalent to $H_0 \in C^1(\mathbf{R}^N \setminus \{0\})$

Let Δ_H be the Finsler-Laplace operator associated with the norm H, that is

$$\Delta_H u := \operatorname{div}(\nabla_\xi V(\nabla u)) = \operatorname{div}(H(\nabla u) \nabla_\xi H(\nabla u)),$$

where $V(\xi) := H(\xi)^2/2$. The Finsler-Laplace operator has been treated by many mathematicians from various points of view. This talk is concerned with the Cauchy problem for the Finsler heat equation

$$\partial_t u = \Delta_H u, \qquad x \in \mathbf{R}^N, \quad t > 0, \tag{F}$$

which is introduced as a gradient flow of the energy

$$I[u] := \frac{1}{2} \int_{\mathbf{R}^N} H(\nabla u)^2 \, dx.$$

We remark that

> if u is a solution of (F), then ku and $u(kx, k^2 t)$ are also solutions of (F) for any $k \in \mathbf{R}$

In this talk we introduce a notion of H_0-radially symmetric functions and show some nice properties of the Finsler-Laplace operators. In particular, we show that the Finsler-Laplace operator acts as a linear operator on H_0-radially symmetric smooth functions. Furthermore, we obtain an optimal growth condition on initial data for the existence of solutions to the Cauchy problem for the Finsler heat equation.

This is a joint work with Goro Akagi (Tohoku University) and Ryuichi Sato (Tohoku University).

References

[1] G. Akagi, K. Ishige and R. Sato, The Cauchy problem for the Finsler heat equation, to appear in Adv. Calc. Var.

[2] D. G. Aronson, Non-negative solutions of linear parabolic equations, Ann. Scuola Norm. Sup. Pisa **22** (1968), 607–694.

[3] A. Cianchi and P. Salani, Overdetermined anisotropic elliptic problems, Math. Ann. **345** (2009), 859–881.

[4] V. Ferone and B. Kawohl, Remarks on a Finsler-Laplacian, Proc. Amer. Math. Soc. **137** (2009), 247–253.

[5] K. Ishige, On the existence of solutions of the Cauchy problem for a doubly nonlinear parabolic equation, SIAM J. Math. Anal. **27** (1996), 1235–1260.